ON AUTOMORPHISMS GROUPS OF STRUCTURES OF COUNTABLE COFINALITY

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Abstract. In [2] Su Gao proves that the following are equivalent for a countable $\mathcal{M}$ (cf. theorem 1.2 too):

(I) There is an uncountable model of the Scott sentence of $\mathcal{M}$.

(II) There exists some $j \in \overline{Aut(\mathcal{M})} \setminus Aut(\mathcal{M})$, where $\overline{Aut(\mathcal{M})}^T$ is the closure of $Aut(\mathcal{M})$ under the product topology in ω^ω.

(III) There is an $\mathcal{L}_{\omega_1,\omega}$- elementary embedding j from $\mathcal{M}$ to itself such that $range(j) \subset \mathcal{M}$.

We generalize his theorem to all cardinalities of cofinality ω (cf. theorem 3.2). In particular, if κ is a cardinal of cofinality ω, then the following are equivalent:

(I*) There is a model of the Scott sentence of $\mathcal{M}$ of size κ^+.

(II*) For all $\alpha < \beta < \kappa^+$, there exist functions $j_{\beta,\alpha}$ in $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$, such that for $\alpha < \beta < \gamma < \kappa^+$,

$$j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}, \qquad (*)$$

where $\overline{Aut(\mathcal{M})}^T$ is the closure of $Aut(\mathcal{M})$ under the product topology in κ^κ.

(III*) For every $\beta < \kappa^+$, there exist $\mathcal{L}^{\text{fin}}_{\infty,\kappa}$- elementary embeddings (cf. definition 2.5) $(j_\alpha)_{\alpha<\beta}$ from M to itself such that $\alpha_1 < \alpha_2 \Rightarrow range(j_{\alpha_1}) \subset range(j_{\alpha_2})$.

Theorem 3.2 holds both for countable and uncountable κ. Condition (*) in (II*), which does not appear in the countable case, can not be removed when κ is uncountable (cf. theorem 3.4).

Combining with a theorem of Kueker (cf. theorem 3.3), conditions (I*), (II*) and (III*) imply the existence of at least κ^ω automorphisms of $\mathcal{M}$. (cf. corollary 3.7).

1. Background

In [2], Su Gao proves the following

Theorem 1.1 (Su Gao). *The following are equivalent for a countable $\mathcal{M}$:*

(I) *There is no uncountable model of the Scott sentence of $\mathcal{M}$.*

(II) *$Aut(\mathcal{M})$ admits a compatible left-invariant complete metric.*

(III) *There is no $\mathcal{L}_{\omega_1,\omega}$- elementary embedding from $\mathcal{M}$ to itself which is not onto.*

2010 *Mathematics Subject Classification.* Primary 03C75, 03C52 Secondary 54H11, 20B27, 20B30.

Key words and phrases. Automorphism Group, Scott sentence, Infinitary Languages, Countable Cofinality.

[0]Author gratefully acknowledges support by the Faculty Grant Incentive Program (FGIP) of the University of Detroit Mercy.

It is also proven in [2] that condition (II) of theorem 1.1 is equivalent to

$$(II)' \qquad Aut(\mathcal{M}) \text{ is closed in the Baire space } \omega^\omega.$$

Using $(II)'$ and taking negations, we rephrase theorem 1.1:

Theorem 1.2 (Su Gao). *The following are equivalent for a countable model $\mathcal{M}$:*

(I) *There is an uncountable model of the Scott sentence of $\mathcal{M}$.*
(II) *There exists some $j \in \overline{Aut(\mathcal{M})} \setminus Aut(\mathcal{M})$, where $\overline{Aut(\mathcal{M})}^T$ is the closure of $Aut(\mathcal{M})$ under the product topology in ω^ω.*
(III) *There is an $\mathcal{L}_{\omega_1,\omega}$- elementary embedding j from $\mathcal{M}$ to itself such that $range(j) \subset \mathcal{M}$.*

Notation: Throughout the whole paper both $\subset$ and $\supset$ refer to *strict* subset and *strict* superset relations.

The main theorem is theorem 3.2 and generalizes theorem 1.2 to any cardinal of cofinality ω. Section 3 is devoted to the proof of the main theorem and its corollaries. Section 2 contains some preliminary work and section 4 contains open questions. The proofs are mainly straightforward.

2. Preliminaries

One obstacle in extending theorem 1.2 to uncountable cardinals is that if $\mathcal{M}$ is a model of uncountable cardinality, there maybe no Scott sentence for $\mathcal{M}$. If $cf(|\mathcal{M}|) = \omega$, then the Scott sentence is guaranteed by the following theorem from [1]:

Theorem 2.1 (C. C. Chang). *Let $\mathcal{N}$ be a model of cardinality κ with $cf(\kappa) = \omega$. Then there is a sentence $\phi_{\mathcal{N}}$ in $\mathcal{L}_{(\kappa^{<\kappa})^+,\kappa}$ such that:*

(1) *For any $\mathcal{N}'$ of cardinality $\leq \kappa$, $\mathcal{N}' \models \phi_N$ iff $\mathcal{N}' \cong \mathcal{N}$.*
(2) *If $\mathcal{N}'$ is any model (possibly of cardinality $> \kappa$), then $\mathcal{N}' \models \phi_N$ iff $\mathcal{N}' \equiv_{\infty,\kappa} \mathcal{N}$.*

For this theorem also follows

Theorem 2.2. *Let $\mathcal{N}$ be a model of cardinality κ with $cf(\kappa) = \omega$ and let N_0 be a subset of $\mathcal{N}$ of size $< \kappa$. Then there is a sentence $\phi_{\mathcal{N}}^{N_0}$ in $\mathcal{L}_{(\kappa^{<\kappa})^+,\kappa}$ such that:*

(1) *If $\mathcal{N}'$ is a model cardinality $\leq \kappa$ and $N_0' \subset \mathcal{N}'$, then $\mathcal{N}' \models \phi_{\mathcal{N}}^{N_0}[N_0']$ iff there is an isomorphism $i : \mathcal{N}' \cong \mathcal{N}$ with $i(N_0) = N_0'$.*
(2) *If $\mathcal{N}'$ is a model of any cardinality (possibly $> \kappa$) and $N_0' \subset \mathcal{N}'$, then $\mathcal{N}' \models \phi_{\mathcal{N}}^{N_0}[N_0']$ iff $(\mathcal{N}', N_0') \equiv_{\infty,\kappa} (\mathcal{N}, N_0)$.*

In particular, the above theorem holds for N_0 finite. The sentence $\phi_{\mathcal{N}}^{N_0}$ is called the Scott sentence of N_0 (in $\mathcal{N}$).

Definition 2.3. *$\mathcal{A}$ and $\mathcal{B}$ are κ- partially isomorphic, write $A \cong_\kappa \mathcal{B}$, if there is a non-empty set I of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ with the κ-back-and-forth property:*

for any $f \in I$ and $C \subset \mathcal{A}$ with $|C| < \kappa$ (or $D \subset \mathcal{B}$ with $|D| < \kappa$),
there is some $g \in I$ that extends f and $C \subset dom(g)$ (or $D \subset range(g)$)

The following two theorems are from [5]. The first is attributed to C. Karp.

Theorem 2.4. *Let $\kappa \geq \omega$. Then for any $\mathcal{N}, \mathcal{N}'$,*

(1) $\mathcal{N}' \equiv_{\infty,\kappa} \mathcal{N}$ *iff* $\mathcal{N}' \cong_\kappa \mathcal{N}$
(2) $\mathcal{N}' \equiv_{\infty,\kappa} \mathcal{N}$ *iff for every* $\vec{a} \in \mathcal{N}'^{<\kappa}$, *there is some* $\vec{b} \in \mathcal{N}^{<\kappa}$ *such that* $(\mathcal{N}', \vec{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \vec{b})$.

Note that in the second part we can switch the roles of $\vec{a}$ *and* $\vec{b}$, *i.e.*

$\mathcal{N}' \equiv_{\infty,\kappa} \mathcal{N}$ *iff for every* $\vec{b} \in \mathcal{N}^{<\kappa}$ *there is some* $\vec{a} \in \mathcal{N}'^{<\kappa}$ *such that* $(\mathcal{N}', \vec{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \vec{b})$.

Definition 2.5. *The* $(\mathcal{L}_{\omega,\omega})$*- embedding* $j : \mathcal{M} \to \mathcal{N}$ *will be called a* $\mathcal{L}^{fin}_{\infty,\kappa}$*- elementary embedding if for every formula* $\phi(\vec{x}) \in \mathcal{L}_{\infty,\kappa}$ *with* finitely many *free variables, and for every finite* $\vec{a} \in M$,

$$\mathcal{M} \models \phi[\vec{a}] \text{ iff } \mathcal{N} \models \phi[j(\vec{a})].$$

Similarly we define $\mathcal{L}^{fin}_{\infty,\kappa}$*- elementary substructures:*

If $\mathcal{M} \subset \mathcal{N}$ *we will call* $\mathcal{M}$ *a* $\mathcal{L}^{fin}_{\infty,\kappa}$*- elementary substructure of* $\mathcal{N}$ *and write* $\mathcal{M} \prec^{fin}_{\infty,\kappa} \mathcal{N}$, *if the inclusion map* $id : \mathcal{M} \to \mathcal{N}$ *is an* $\mathcal{L}^{fin}_{\infty,\kappa}$*- elementary embedding.*

The difference than the regular definition of $\mathcal{L}_{\infty,\kappa}$- elementary embedding is that we restrict ourselves to finite $\vec{a}$ only. The motivation for this definition is from lemma 2.7. Observe also that if $\kappa > \omega$, the $\mathcal{L}_{\infty,\kappa}$- formulas with finitely many free variables are not closed under subformulas.

Lemma 2.6. *If* $\mathcal{M}_0, \mathcal{M}_1 \prec^{fin}_{\infty,\kappa} \mathcal{M}_2$ *and* $\mathcal{M}_0 \subset \mathcal{M}_1$, *then* $\mathcal{M}_0 \prec^{fin}_{\infty,\kappa} \mathcal{M}_1$.

Proof. As in the first-order case. □

Lemma 2.7. *Let* κ *be a cardinal of cofinality* ω *and* $\mathcal{M}$ *a model of size* κ. *If* $\overline{Aut(\mathcal{M})}^T$ *is the closure of* $Aut(\mathcal{M})$ *in the product topology* T, *then* $j \in \overline{Aut(\mathcal{M})}^T$ *iff* j *is an* $\mathcal{L}^{fin}_{\infty,\kappa}$*- elementary embedding from* $\mathcal{M}$ *to itself.*

Proof. Let j be in $\overline{Aut(\mathcal{M})}^T$ and $\vec{a} \in \mathcal{M}^{<\omega}$. By the definition of the topology, there must be an automorphism $f \in Aut(\mathcal{M})$ such that,

$$f(\vec{a}) = j(\vec{a}).$$

Then $\phi^{\vec{a}}_{\mathcal{M}} = \phi^{f(\vec{a})}_{\mathcal{M}} = \phi^{j(\vec{a})}_{\mathcal{M}}$, which proves that j is an elementary $\mathcal{L}^{\text{fin}}_{\infty,\kappa}$- embedding from $\mathcal{M}$ to $\mathcal{M}$.

Conversely, assume that $j : \mathcal{M} \to \mathcal{M}$ is an $\mathcal{L}^{\text{fin}}_{\infty,\kappa}$- elementary embedding. In particular, j is an isomorphism between $\mathcal{M}$ and $j[\mathcal{M}]$, and if $\vec{a} \in \mathcal{M}^{<\omega}$, then

$$j[\mathcal{M}] \models \phi^{\vec{a}}_{\mathcal{M}}[j(\vec{a})].$$

By elementarity,

$$\mathcal{M} \models \phi^{\vec{a}}_{\mathcal{M}}[j(\vec{a})].$$

By theorem 2.2, there is an automorphism f of $\mathcal{M}$ such that $f(\vec{a}) = j(\vec{a})$. Since this is true for any $\vec{a}$, j is in the closure of $Aut(\mathcal{M})$ in the product topology T. □

The following theorem will be used in place of the Downward Lowenheim- Skolem theorem:

Theorem 2.8. *Let κ be a cardinal of cofinality ω. Let $\mathcal{N}$ be a structure of cardinality κ and $\phi_{\mathcal{N}}$ its Scott sentence. If $\mathcal{A}$ is a model of $\phi_{\mathcal{N}}$ of any size (possibly $> \kappa$) and A_0 is a subset of $\mathcal{A}$ of size $\leq \kappa$, then there is some $A_1 \subset \mathcal{A}$ such that $A_0 \subset A_1$, there exists some isomorphism $i : A_1 \cong \mathcal{N}$ and $A_1 \prec^{fin}_{\infty,\kappa} \mathcal{A}$.*

Proof. By Theorem 2.2, $\mathcal{A}$ and $\mathcal{N}$ are $\equiv_{\infty,\kappa}$- equivalent.

Since κ has cofinality ω, assume that $\kappa = \cup_n \kappa_n$. Then we can write A_0 as the union of $A_{0,n}$, for $n < \omega$, where $|A_{0,n}| = \kappa_n$. Similarly we can write $\mathcal{N}$ as the union of $\mathcal{N}_n$ where $|\mathcal{N}_n| = \kappa_n$.

Using Theorem 2.4, part (2), we can give the usual back-and-forth argument to construct subsets $M_n \subset \mathcal{N}$ and $B_n \subset \mathcal{A}$, for all $n \in \omega$, with the following properties:

(1) $M_n \subset M_m$, for $n < m$,
(2) $B_n \subset B_m$, for $n < m$,
(3) $M_{2n} \supset N_n$,
(4) $B_{2n+1} \supset A_{0,n}$ and
(5) $(\mathcal{N}, M_n) \equiv_{\infty,\kappa} (\mathcal{A}, B_n)$.

By Theorem 2.4, it also follows that $(\mathcal{N}, M_n) \cong_\kappa (\mathcal{A}, B_n)$, i.e. there is a partial isomorphism $i_n : M_n \to B_n$.

Taking unions $A_1 = \cup_n B_n$ and $i = \cup_n i_n$, we have $A_1 \subset \mathcal{A}$, $A_1 \supset A_0$ and i is an isomorphism between $\mathcal{N}$ and A_1. Since every finite $\vec{a} \in \mathcal{N}$ will be included in some M_n, it follows by Theorem 2.2 that $\phi^{\vec{a}}_{\mathcal{N}} = \phi^{i(\vec{a})}_{\mathcal{A}}$. Since $\phi^{i(\vec{a})}_{A_1} = \phi^{\vec{a}}_{\mathcal{N}}$, the result follows. □

Notice that the proof works only for if we restrict ourselves to *finite* $\vec{a} \in \mathcal{N}$. Thus, Theorem 2.8 provides a second justification for the use of $\mathcal{L}^{\text{fin}}_{\infty,\kappa}$.

Lemma 2.9 (Trevor Wilson[1])**.** *If $|\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})| \geq 1$, then $|\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})| \geq |Aut(M)|$.*

Proof. Let $f \in \overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$. Then $g \mapsto f \circ g$ is an injection from $Aut(M)$ to $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$. □

Lemma 2.10. *Let κ be a cardinal of cofinality ω and $\mathcal{M}$ a model of size κ. The following are equivalent:*

(1) *There is a strictly increasing chain of models $(\mathcal{M}_\alpha)_{\alpha<\kappa^+}$ of cardinality κ such that $\mathcal{M}_0 = \mathcal{M}$ and for all $\alpha < \beta$, $\mathcal{M}_\alpha \prec^{fin}_{\infty,\kappa} \mathcal{M}_\beta$.*
(2) *There is a model of the Scott sentence of $\mathcal{M}$ of size κ^+.*

Proof. (2)⇒ (1). Assume that $\mathcal{N}$ is a model of $\phi_{\mathcal{M}}$ of size κ^+, where $\phi_{\mathcal{M}}$ is the Scott sentence of $\mathcal{M}$. Construct an increasing chain of models $(\mathcal{M}_\alpha)_{\alpha<\kappa^+}$ by induction. Assume that for some $\alpha < \kappa^+$, there exists a chain $(\mathcal{M}_\gamma)_{\gamma<\alpha}$ such that $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_\gamma \cong \mathcal{M}$ and $\mathcal{M}_\gamma \prec^{\text{fin}}_{\infty,\kappa} \mathcal{N}$, for all $\gamma < \alpha$. Extend this sequence to $\mathcal{M}_\alpha$ and the construction works for both α successor and α limit ordinal. Let $U = \cup_{\gamma<\alpha} \mathcal{M}_\gamma$ and U has cardinality κ. Then there exists some $a \in \mathcal{N} \setminus U$ and apply theorem 2.8 to find some $\mathcal{M}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \mathcal{N}$ which contains $U \cup \{a\}$ and is isomorphic to $\mathcal{M}$. It follows from lemma 2.6 that for all $\gamma < \alpha$, $\mathcal{M}_\gamma \prec^{\text{fin}}_{\infty,\kappa} \mathcal{M}_\alpha$.

[1]The proof of this corollary is due to Trevor Wilson who answered a corresponding question on MathOverflow. See [6]

(1)$\Rightarrow$ (2). The argument here is essentially the argument that proves that $\mathcal{L}_{\infty,\kappa}$ is closed under unions of size κ^+ (cf. [5]).

Assume that there exists an increasing chain $(\mathcal{M}_\alpha)_{\alpha<\kappa^+}$ as in (1). Let $\mathcal{N} = \cup_{\alpha<\kappa^+}\mathcal{M}_\alpha$. We claim that for all $\alpha < \kappa^+$, $\mathcal{M}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \mathcal{N}$. Let $\vec{a}$ be some (finite) tuple in $\mathcal{M}_\alpha$. We prove by induction on $\phi \in \mathcal{L}_{\infty,\kappa}$ that

$$\mathcal{M}_\alpha \models \phi[\vec{a}] \text{ iff } \mathcal{N} \models \phi[\vec{a}].$$

If ϕ is atomic the result follows from $\mathcal{M}_\alpha$ being a substructure of $\mathcal{N}$. The cases where ϕ is a conjunction or disjunction are immediate. We prove the case where ϕ is of the form $\exists X\psi[\vec{a}, X]$, where X is a subset of size $< \kappa$. The case $\forall X\psi[\vec{a}, X]$ is proved similarly. So, assume that $\mathcal{N} \models \exists X\psi[\vec{a}, X]$ and fix some X of size $< \kappa$ so that $\mathcal{N} \models \psi[\vec{a}, X]$. By cardinality considerations there must be some $\beta < \kappa^+$ so that $\beta \geq \alpha$ and $X \subset \mathcal{M}_\beta$. By the induction hypothesis, $\mathcal{M}_\beta \models \psi[\vec{a}, X]$, i.e. $\mathcal{M}_\beta \models \exists X\psi[\vec{a}, X]$. Since $\mathcal{M}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \mathcal{M}_\beta$, it follows that also $\mathcal{M}_\alpha \models \exists X\psi[\vec{a}, X]$, i.e. $\mathcal{M}_\alpha \models \phi[\vec{a}]$. The left-to-right direction is immediate. $\square$

Definition 2.11. *Let*

$$\mathcal{M}_0 \prec^{fin}_{\infty,\kappa} \mathcal{M}_1 \prec^{fin}_{\infty,\kappa} \dots \prec^{fin}_{\infty,\kappa} \mathcal{M}_\beta$$

and

$$\mathcal{N}_0 \prec^{fin}_{\infty,\kappa} \mathcal{N}_1 \prec^{fin}_{\infty,\kappa} \dots \prec^{fin}_{\infty,\kappa} \mathcal{N}_\gamma$$

be two $\mathcal{L}^{fin}_{\infty,\kappa}$- increasing sequences. Notice that the lengths of the two sequences, β and γ, may not be the same.

We call the two sequences compatible *if there is an isomorphism that maps the one sequence to an initial segment of the other. E.g. if $\beta < \gamma$, the two sequences are* compatible *if there is an isomorphism $i : \mathcal{M}_\beta \cong \mathcal{N}_\beta$ such that $i[\mathcal{M}_\alpha] = \mathcal{N}_\alpha$ for all $\alpha < \beta$. Similarly for the cases $\gamma < \beta$ and $\gamma = \beta$.*

The following two lemmas are immediate

Lemma 2.12. *Compatibility of sequences is a transitive relation.*

Lemma 2.13. *Let $\beta < \gamma$ and $(\mathcal{M}_\alpha)_{\alpha\leq\beta}$ and $(\mathcal{N}_\alpha)_{\alpha\leq\gamma}$ be two compatible sequences as witnessed by $i : \mathcal{M}_\beta \cong \mathcal{N}_\beta$. Then $(\mathcal{M}_\alpha)_{\alpha\leq\beta}$ can be extended to $(\mathcal{M}_\alpha)_{\alpha\leq\gamma}$ in such a way that there exists some $i' : \mathcal{M}_\gamma \cong \mathcal{N}_\gamma$, $i' \supset i$ and $i'[\mathcal{M}_\alpha] = \mathcal{N}_\alpha$ for all $\alpha < \gamma$, i.e. i' witnesses the fact that $(\mathcal{M}_\alpha)_{\alpha\leq\gamma}$ and $(\mathcal{N}_\alpha)_{\alpha\leq\gamma}$ are compatible.*

Lemma 2.14. *Let κ be a cardinal of cofinality ω and assume that for every $\beta < \kappa^+$ there exist $\prec^{fin}_{\infty,\kappa}$- increasing sequences $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$ all of which are compatible with each other. Then there exists a sequence $(\mathcal{M}_\alpha)_{\alpha<\kappa^+}$ of length κ^+ which is compatible with all sequences $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$.*

Proof. Proceed by induction on $\gamma < \kappa^+$. Assume that $(M_\alpha)_{\alpha<\gamma}$ is given and is compatible with all $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$. Fix some $\beta > \gamma$ and use lemma 2.13 to extend $(M_\alpha)_{\alpha<\gamma}$ to $(M_\alpha)_{\alpha<\beta}$ which is compatible with $(M^\beta_\alpha)_{\alpha<\beta}$. The details follow. $\square$

The next few lemmas are mainly about renaming and the proofs are straightforward. Nevertheless, the results will be needed later on in the proof of the main theorem 3.2. We present the proof of only one of the lemmas and the other proofs follow using similar arguments.

Lemma 2.15. *Let κ be a cardinal of cofinality ω and $\mathcal{M}$ be a model of size κ. Assume that*

$$\mathcal{M}_0 \prec^{fin}_{\infty,\kappa} \mathcal{M}_1 \prec^{fin}_{\infty,\kappa} \mathcal{M}_2 = \mathcal{M}.$$

Then there exists a sequence

$$\mathcal{N}_0 = \mathcal{M} \prec^{fin}_{\infty,\kappa} \mathcal{N}_1 \prec^{fin}_{\infty,\kappa} \mathcal{N}_2$$

which is compatible with $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$.

Proof. By theorem 2.1 $\mathcal{M}_0$ is isomorphic to $\mathcal{M}$ and let i be such an isomorphism. Take $\mathcal{M}'_2$ to be an isomorphic copy of $\mathcal{M}_2$ such that $\mathcal{M}'_2 \cap \mathcal{M}_2 = \mathcal{M}_0$ and let $\mathcal{M}'_1 \subset \mathcal{M}'_2$ be the isomorphic image of $\mathcal{M}_1$ inside $\mathcal{M}'_2$. Let $\mathcal{N}_0 = \mathcal{M}$, $\mathcal{N}_1 = \mathcal{M} \cup (\mathcal{M}'_1 \setminus \mathcal{M}_0)$, $\mathcal{N}_2 = \mathcal{M} \cup (\mathcal{M}'_2 \setminus \mathcal{M}_0)$ and let $j : \mathcal{N}_2 \to \mathcal{M}'_2$ be the function

$$j(x) = \begin{cases} x & \text{if } x \in \mathcal{M}'_2 \setminus \mathcal{M}_0 \ , \\ i(x) & \text{if } x \in \mathcal{M}. \end{cases}$$

Define

$$\mathcal{N}_i \models \phi[\vec{x}] \text{ iff } \mathcal{M}'_i \models \phi[j(\vec{x})], \text{ for } i = 1, 2.$$

It follows by the definitions that j is an isomorphism between $\mathcal{N}_2$ and $\mathcal{M}'_2$ such that $j \restriction_{\mathcal{N}_1} : \mathcal{N}_1 \cong \mathcal{M}'_1$, $j \restriction_{\mathcal{N}_0} = i : \mathcal{N}_0 \cong \mathcal{M}_0$ and that $\mathcal{N}_0 = \mathcal{M} \prec^{\text{fin}}_{\infty,\kappa} \mathcal{N}_1 \prec^{\text{fin}}_{\infty,\kappa} \mathcal{N}_2$. We complete the proof using the isomorphism of $\mathcal{M}_2$ and $\mathcal{M}'_2$. $\square$

Corollary 2.16. *Let κ be a cardinal of cofinality ω, $\mathcal{M}$ be a model of size κ and fix some ordinal $\beta < \kappa^+$. Assume that $(\mathcal{M}_\alpha)_{\alpha \leq \beta}$ is an $\mathcal{L}^{fin}_{\infty,\kappa}$- increasing chain of models and $\mathcal{M}_\beta = \mathcal{M}$. Then there exists another $\mathcal{L}^{fin}_{\infty,\kappa}$- increasing chain $(\mathcal{N}_\alpha)_{\alpha \leq \beta}$ that is compatible with $(\mathcal{M}_\alpha)_{\alpha \leq \beta}$ and $\mathcal{N}_0 = \mathcal{M}$.*

Corollary 2.17. *Let κ, $\mathcal{M}$ and β be as in the previous corollary and let γ be some ordinal $< \beta$. Assume that $(\mathcal{M}_\alpha)_{\alpha \leq \beta}$ is an $\mathcal{L}^{fin}_{\infty,\kappa}$- increasing chain of models, all of which have cardinality κ, and there exists some $\alpha \leq \beta$ such that $\mathcal{M}_\alpha = \mathcal{M}$. Then there exists another $\mathcal{L}^{fin}_{\infty,\kappa}$- increasing chain $(\mathcal{N}_\alpha)_{\alpha \leq \beta}$ that is compatible with $(\mathcal{M}_\alpha)_{\alpha \leq \beta}$ and $\mathcal{N}_\gamma = \mathcal{M}$.*

Definition 2.18. *Let κ be an infinite cardinal and $\mathcal{M}$ a structure of size κ. $\mathcal{M}$ will be called κ-* homogeneous *if every isomorphism between substructures of $\mathcal{M}$ generated by $(< \kappa)$- many elements, can extend to an automorphism of $\mathcal{M}$.*

Theorem 2.19. *Let $\mathcal{M}$ be a model of size κ and $\mathcal{N} \supset \mathcal{M}$ an ω-homogeneous model (possibly of size $> \kappa$). Then $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ iff $\mathcal{M} \prec^{fin}_{\infty,\kappa} \mathcal{N}$.*

Proof. The right-to-left implication is immediate. So, assume that $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$. Let $\vec{a} \in \mathcal{M}^{<\omega}$ and let $\phi^{\vec{a}}_{\mathcal{M}} \in \mathcal{L}_{\infty,\kappa}$ be the Scott sentence of $\vec{a}$ in $\mathcal{M}$. We must prove that $\mathcal{N} \models \phi^{\vec{a}}_{\mathcal{M}}[\vec{a}]$. By theorem 2.4, there exists some $\vec{b} \in \mathcal{N}^{<\omega}$ such that $(\mathcal{M}, \vec{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \vec{b})$. In particular, $\mathcal{N} \models \phi^{\vec{a}}_{\mathcal{M}}[\vec{b}]$ and there exists an isomorphism p between $\vec{a}$ and $\vec{b}$. By $\mathcal{N}$ being ω- homogeneous, p can be extended to some automorphism j of $\mathcal{N}$ so that $j[\vec{a}] = \vec{b}$. It follows that $\mathcal{N} \models \phi^{\vec{a}}_{\mathcal{M}}[\vec{a}]$ as desired. $\square$

We now are ready to generalize theorem 1.2 to all cardinalities of cofinality ω.

3. Main Theorem

Recall theorem 1.2

Theorem 3.1 (Su Gao)**.** *The following are equivalent for a countable model $\mathcal{M}$:*

(I) *There is an uncountable model of the Scott sentence of $\mathcal{M}$.*

(II) *There exists some $j \in \overline{Aut(\mathcal{M})} \setminus Aut(\mathcal{M})$, where $\overline{Aut(\mathcal{M})}^T$ is the closure of $Aut(\mathcal{M})$ under the product topology in ω^ω.*

(III) *There is an $\mathcal{L}_{\omega_1,\omega}$- elementary embedding j from $\mathcal{M}$ to itself such that $range(j) \subset \mathcal{M}$.*

We prove the following:

Theorem 3.2. *Let κ be an uncountable cardinal of cofinality ω and $\mathcal{M}$ a model of size κ. The following are equivalent:*

(I*) *There is a model of the Scott sentence of $\mathcal{M}$ of size κ^+.*

(II*) *For all $\alpha < \beta < \kappa^+$, there exist functions $j_{\beta,\alpha}$ in $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$, such that for $\alpha < \beta < \gamma < \kappa^+$,*

$$j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}, \tag{*}$$

where $\overline{Aut(\mathcal{M})}^T$ is the closure of $Aut(\mathcal{M})$ under the product topology in κ^κ.

(III*) *For every $\beta < \kappa^+$, there exist $\mathcal{L}^{fin}_{\infty,\kappa}$- elementary embeddings $(j_\alpha)_{\alpha<\beta}$ from M to itself such that $\alpha_1 < \alpha_2 \Rightarrow range(j_{\alpha_1}) \subset range(j_{\alpha_2})$.*

We will prove the main theorem in two steps. The equivalence of (I*) and (II*) is by theorem 3.5 and the equivalence of (I*) and (III*) by theorem 3.6.

A couple of notes before we proceed: Although theorem 3.2 holds true for both κ countable and uncountable, it is of interest in the uncountable case. If κ is countable, theorem 1.2 provides sharper equivalent conditions. On the other hand, if κ is an uncountable cardinal, (*) can not be omitted from (II*). I.e. mere existence of κ^+ many elements in $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$, or even existence of κ^ω many such elements, is not sufficient to prove the existence of a model in κ^+. This is proved in theorem 3.4 and the argument is based on the following theorem of Kueker (cf. [5],[4]).

Theorem 3.3 (Kueker)**.** *Let κ be an uncountable cardinal of cofinality ω and $\mathcal{M}$ a model of size κ. Then the following implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold true, but there exist counterexamples for the inverse implications:*

(1) *There is a model $\mathcal{N}$ of size $> \kappa$ such that $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$.*

(2) *For every $\vec{a} \in \mathcal{M}^{<\kappa}$, $(\mathcal{M}, \vec{a})$ has a proper automorphism.*

(3) *$\mathcal{M}$ has at least κ^ω automorphisms.*

Theorem 3.4. *Let κ be an uncountable cardinal of cofinality ω. Then there exists a model $\mathcal{M}$ of size κ such that:*

(1) *there is no model of the Scott sentence of $\mathcal{M}$ of size κ^+*

(2) *there exists at least κ^ω many elements in $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$.*

Proof. By theorem 3.3 and lemma 2.9. □

Theorem 3.5. *Let κ be a cardinal of cofinality ω and $\mathcal{M}$ a model of size κ. The following are equivalent:*

(I*) *There is a model of the Scott sentence of $\mathcal{M}$ of size κ^+.*

(II*) *For all $\alpha < \beta < \kappa^+$, there exist functions $j_{\beta,\alpha}$ in $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$, such that for $\alpha < \beta < \gamma < \kappa^+$,*

$$j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}, \tag{*}$$

where $\overline{Aut(\mathcal{M})}^T$ is the closure of $Aut(\mathcal{M})$ under the product topology in κ^κ.

Proof. By lemma 2.10, (I*) is equivalent to the existence of an increasing chain of models $(\mathcal{M}_\alpha)_{\alpha<\kappa^+}$ of cardinality κ such that $\mathcal{M}_0 = \mathcal{M}$ and for all $\alpha < \beta$, $\mathcal{M}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \mathcal{M}_\beta$. We will work with the latter condition instead of (I*).

(I*)$\Rightarrow$ (II*). Assume the existence of a chain of models $(\mathcal{M}_\alpha)_{\alpha<\kappa^+}$ as given by lemma 2.10. First observe that for all $\alpha < \kappa^+$, $\mathcal{M}_0 = \mathcal{M} \equiv_{\infty,\kappa} \mathcal{M}_\alpha$ implies $\mathcal{M} \cong \mathcal{M}_\alpha$. Thus, we can find isomorphisms $i_\alpha : \mathcal{M}_\alpha \cong \mathcal{M}$, for each $\alpha < \kappa^+$. Let

$$j_{\beta,\alpha} = i_\beta \circ i_\alpha^{-1},$$

for all $\alpha < \beta < \kappa^+$. This is a well-defined function form $\mathcal{M}$ to $\mathcal{M}$ since $\alpha < \beta$ implies $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$, but $j_{\beta,\alpha}$ fails to be onto. On the other hand, $\mathcal{M}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \mathcal{M}_\beta$ implies that $j_{\beta,\alpha}$ is an $\mathcal{L}^{\text{fin}}_{\infty,\kappa}$- elementary embedding from $\mathcal{M}$ to itself. By lemma 2.7, $j_{\beta,\alpha}$ is in $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$ and it remains to prove that for $\alpha < \beta < \gamma < \kappa^+$,

$$j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}.$$

This follows immediately by the definition:

$$\begin{aligned} j_{\gamma,\beta} \circ j_{\beta,\alpha} &= i_\gamma \circ i_\beta^{-1} \circ i_\beta \circ i_\alpha^{-1} \\ &= i_\gamma \circ i_\alpha^{-1} \\ &= j_{\gamma,\alpha}. \end{aligned}$$

(II*)$\Rightarrow$ (I*) Assume the existence of $j_{\beta,\alpha}$'s as in (II*). First observe that by lemma 2.7, every $j_{\beta,\alpha}$ is one-to-one but not onto.

We claim that $range(j_{\gamma,\beta}) \supset range(j_{\gamma,\alpha})$, whenever $\alpha < \beta < \gamma$. By (*), it is obvious that $range(j_{\gamma,\alpha})$ is a subset of $range(j_{\gamma,\beta})$ and since $j_{\beta,\alpha}$ is not onto, they can not be equal.

Next let $\mathcal{M}^\beta_\alpha = j_{\beta,\alpha}[\mathcal{M}]$, for $\alpha < \beta < \kappa^+$, and also let $\mathcal{M}^\beta_\beta = \mathcal{M}$. By definition, $\mathcal{M}^\beta_\alpha \prec^{\text{fin}}_{\infty,\kappa} \mathcal{M}^\beta_\beta$, for all $\alpha < \beta$. We also claim that for every $\beta, \gamma < \kappa^+$, the sequences $(\mathcal{M}^\beta_\alpha)_{\alpha\le\beta}$ and $(\mathcal{M}^\gamma_\alpha)_{\alpha\le\gamma}$ are compatible (cf. definition 2.11). Without loss of generality assume that $\beta < \gamma$ and work similarly in the other cases. The isomorphism between $\mathcal{M}^\beta_\beta$ and $\mathcal{M}^\gamma_\beta$ is witnessed by $j_{\gamma,\beta}$:

$$j_{\gamma,\beta}[\mathcal{M}^\beta_\beta] = j_{\gamma,\beta}[\mathcal{M}] = \mathcal{M}^\gamma_\beta.$$

For $\alpha < \beta$ we get

$$\begin{aligned} j_{\gamma,\beta}[\mathcal{M}^\beta_\alpha] &= j_{\gamma,\beta}[j_{\beta,\alpha}[\mathcal{M}]] \\ &= j_{\gamma,\alpha}[\mathcal{M}] \\ &= \mathcal{M}^\gamma_\alpha, \end{aligned}$$

which proves that the sequences $(\mathcal{M}^\beta_\alpha)_{\alpha\le\beta}$ and $(\mathcal{M}^\gamma_\alpha)_{\alpha\le\gamma}$ are compatible.

Here notice for each $\beta < \kappa^+$, the sequences $(\mathcal{M}^\beta_\alpha)_{\alpha\le\beta}$ consist of substructures of $\mathcal{M}$, but using corollary 2.16 we can find $(\mathcal{N}^\beta_\alpha)_{\alpha\le\beta}$ compatible with $(\mathcal{M}^\beta_\alpha)_{\alpha\le\beta}$ such

that $\mathcal{N}_0^\beta = \mathcal{M}$. In particular, the sequences $(\mathcal{N}_\alpha^\beta)_{\alpha\leq\beta}$ and $(\mathcal{N}_\alpha^\gamma)_{\alpha\leq\gamma}$ are compatible with each other for all $\beta, \gamma < \kappa^+$ and we use lemma 2.14 to complete the proof. □

Theorem 3.6. *Let κ be a cardinal of cofinality ω and $\mathcal{M}$ a model of size κ. The following are equivalent:*

(I*) *There is a model of the Scott sentence of $\mathcal{M}$ of size κ^+.*

(III*) *For every $\beta < \kappa^+$, there exist $\mathcal{L}^{fin}_{\infty,\kappa}$- elementary embeddings $(j_\alpha)_{\alpha<\beta}$ from M to itself such that $\alpha_1 < \alpha_2 \Rightarrow range(j_{\alpha_1}) \subset range(j_{\alpha_2})$.*

Proof. (I*)⇒ (III*). By theorem 3.5, (I*) is equivalent to the existence of functions $j_{\beta,\alpha} : \mathcal{M} \to \mathcal{M}$ in $\overline{Aut(\mathcal{M})}^T \setminus Aut(\mathcal{M})$ such that for any $\alpha < \beta < \gamma < \kappa^+$,

$$(*) \qquad j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}$$

As pointed out in the proof of the theorem 3.5 too, (*) implies that $\alpha_1 < \alpha_2 \Rightarrow range(j_{\beta,\alpha_1}) \subset range(j_{\beta,\alpha_2})$.

(III*)⇒ (I*) By lemma 2.10, it suffices to prove that there is a strictly increasing $\prec^{\text{fin}}_{\infty,\kappa}$- chain of models $(\mathcal{M}_\alpha)_{\alpha<\kappa^+}$ of cardinality κ and $\mathcal{M}_0 = M$. By lemma 2.14 it further suffices to prove that for every $\beta < \kappa^+$ there exist $\prec^{\text{fin}}_{\infty,\kappa}$- increasing sequences $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$ all of which are compatible with each other.

We proceed by induction: Assume $\beta < \kappa^+$ and there exists a strictly increasing $\prec^{\text{fin}}_{\infty,\kappa}$- chain of models $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$ of length β. We have to find a compatible chain of length $> \beta$. The idea is to find a $\prec^{\text{fin}}_{\infty,\kappa}$- chain of length β and put it on top of $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$. This will give us the result.

First, by (III*), there are some $\mathcal{L}^{\text{fin}}_{\infty,\kappa}$- elementary embeddings $(j_\alpha)_{\alpha\leq\beta}$ such that their ranges form an increasing sequence under inclusion that has order type $\beta+1$. Let $\mathcal{R}_\alpha = range(j_\alpha)$. Using corollary 2.16 we find a sequence $(\mathcal{S}_\alpha)_{\alpha\leq\beta}$ compatible with $\mathcal{R}_\alpha$ such that $\mathcal{S}_0 = \mathcal{M}$.

On the other hand, using corollary 2.17, we can find a sequence $(\mathcal{N}_\alpha)_{\alpha<\beta}$ such that $\mathcal{N}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \mathcal{M}$ and which is compatible with $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$ (this is the sequence given by the inductive hypothesis).

Putting $(\mathcal{S}_\alpha)_{\alpha\leq\beta}$ on top of $(\mathcal{N}_\alpha)_{\alpha<\beta}$ we get the following sequence which we denote by $(\mathcal{N}_\alpha)_{\alpha<\beta} + (\mathcal{S}_\alpha)_{\alpha\leq\beta}$.

$$\mathcal{N}_0 \prec^{\text{fin}}_{\infty,\kappa} \dots \mathcal{N}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \dots \mathcal{M} = \\ \mathcal{S}_0 \prec^{\text{fin}}_{\infty,\kappa} \dots \mathcal{S}_\alpha \prec^{\text{fin}}_{\infty,\kappa} \dots \mathcal{S}_\beta.$$

The order type of $(\mathcal{N}_\alpha)_{\alpha<\beta} + (\mathcal{S}_\alpha)_{\alpha\leq\beta}$ is equal to $\beta + (\beta + 1) = \beta\cdot 2 + 1 \geq \beta$. Using corollary 2.17 for a second time, we can find a sequence $(\mathcal{M}^{\beta\cdot 2}_\alpha)_{\alpha\leq\beta\cdot 2}$ such that $\mathcal{M}^{\beta\cdot 2}_0 = \mathcal{M}$ and which is compatible with $(\mathcal{N}_\alpha)_{\alpha<\beta} + (\mathcal{S}_\alpha)_{\alpha\leq\beta}$. By assumption $(\mathcal{N}_\alpha)_{\alpha<\beta}$ is compatible with $(\mathcal{M}^\beta_\alpha)_{\alpha<\beta}$. Therefore the same is true for $(\mathcal{M}^{\beta\cdot 2}_\alpha)_{\alpha\leq\beta\cdot 2}$, which finishes the proof. □

Corollary 3.7. *Under the assumptions of theorem 3.2, conditions (I*), (II*) and (III*) imply the existence of κ^ω many automorphisms of $\mathcal{M}$.*

Proof. By theorem 3.3. □

Corollary 3.8. *If in the statement of theorem 3.2 we add the assumption that $\mathcal{M}$ is an ω- homogeneous model, then condition (III*) can be relaxed to the following:*

(III*)′ *For every $\beta < \kappa^+$, there exist $(\mathcal{M}_\alpha)_{\alpha<\beta}$ such that $\alpha_1 < \alpha_2 \Rightarrow \mathcal{M}_{\alpha_1} \subset \mathcal{M}_{\alpha_2} \subset \mathcal{M}$ and every $\mathcal{M}_\alpha$ satisfy the Scott sentence of $\mathcal{M}$.*

Proof. By theorem 2.1 $\mathcal{M}_\alpha \equiv_{\infty,\kappa} \mathcal{M}$. Using ω- homogeneity and theorem 2.19, (III*)′ is equivalent to (III*), which finishes the proof. □

4. Open Questions

We mention some open questions relating to the results in this paper, or extensions of them.

Open Question 1. *Can the results of this paper extend to the case where κ is a successor cardinal? It seems that Ehrenfeucht-Fraisse games, or equivalently the infinitely deep languages $M_{\kappa^+,\kappa}$, must be used instead of $\mathcal{L}_{(\kappa^{<\kappa})^+,\kappa}$.*

Open Question 2. *Can we prove corollary 3.7 directly, without using theorem 3.3?*

Let κ be an infinite cardinal endowed with the discrete topology and let κ^κ be endowed with the product topology. If S_κ denotes the set of 1-1 and onto functions from κ to κ, then S_κ becomes a topological group under composition. If $\mathcal{M}$ is a model of size κ, then it follows by lemma 2.7, $Aut(\mathcal{M})$ is a closed subgroup of S_κ.

The following property is inspired by (II*).

Definition 4.1. *Let G be a closed subgroup of S_κ. We say that G has* large closure *if for all $\alpha < \beta < \gamma < \kappa^+$ there exist $j_{\beta,\alpha} \in \overline{G} \setminus G$ such that $j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}$, where $\overline{G}$ is the closure of G in κ^κ under the product topology.*

Lemma 4.2. *S_κ has large closure.*

Proof. Let α be an ordinal such that $\kappa \leq \alpha < \kappa^+$. Let i_α be a bijection from κ to α. Let $j_{\beta,\alpha} = i_\beta^{-1} \circ i_\alpha$, where $\alpha < \beta$. The reader can verify that these $j_{\beta,\alpha}$'s are 1-1, but not onto functions, and they witness the large closure of S_κ. □

The following two open questions are motivated by similar results in [2] for κ countable.

Open Question 3. *Let G be a closed subgroup of S_κ and there exists a continuous onto homomorphism $p : G \to S_\kappa$. Can we conclude that G has large closure?*

Open Question 4. *Let G be a closed subgroup of S_κ and let H be a closed normal subgroup of G. Then G has large closure iff H has large closure or G/H has large closure.*

An positive answer to open question 4 implies a positive answer to open question 3. Indeed, since $S_\kappa = p(G)$ has large closure, the same is true for $G/Ker(p)$ and by a positive answer to question 4, the same is true for G.

If κ is countable, both questions 3 and 4 have positive answers as proved in [2]. In [3], Hjorth used these positive answers to prove theorem 4.4. We need a definition before we can state the theorem.

Definition 4.3. *Let P be a unary predicate and let $\mathcal{M}$ be a model in a language that contains P. Then P is* homogeneous *for $\mathcal{M}$, if $P(\mathcal{M})$ is infinite and every permutation of it extends to an automorphism of $\mathcal{M}$.*

Theorem 4.4 (Hjorth)**.** *Let $\mathcal{M}$ be a countable model in a language that contains a unary predicate P. If P is homogeneous for $\mathcal{M}$, then the Scott sentence of $\mathcal{M}$ has a model of size $\aleph_1$.*

Assuming an affirmative answer to open question 3, we can generalize theorem 4.4 to uncountable structures of cofinality ω.

Theorem 4.5. *Assume an affirmative answer to open question 3. Let κ be an infinite cardinal of cofinality ω and let $\mathcal{M}$ be a model of size κ in a language that contains a unary predicate P. If P is homogeneous for $\mathcal{M}$ and $P(\mathcal{M})$ has size κ, then the Scott sentence of $\mathcal{M}$ has a model of size κ^+.*

Proof. Let $\sigma \in Aut(\mathcal{M})$. Since $P(\mathcal{M})$ has size κ, we can assume that $P(\mathcal{M}) = \kappa$ and consider $\sigma|_{P(\mathcal{M})}$ as a function from κ to κ. By the homogeneity of P, the map $\sigma \mapsto \sigma|_{P(\mathcal{M})}$ from $Aut(\mathcal{M})$ to S_κ is onto and the reader can verify that it is also a continuous homomorphism. Assuming an affirmative answer to open question 3, $Aut(\mathcal{M})$ has large closure, which by theorem 3.2 implies that the Scott sentence of $\mathcal{M}$ has a model of size κ^+. $\square$

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